ISOVARIANT MAPS FROM FREE G-MANIFOLDS TO REPRESENTATION SPHERES (Transformation groups from a new viewpoint)

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ABSTRACT. The notion of an isovariant map, i.e, an equivariant map preserving the isotropy subgroups, plays an important role in equivariant topology. In this article, we shall formulate the isovariant version of Hopf's classification theorem using the notion of the multidegree. This work is joint with F. Ushitaki.

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1. BACKGROUND — HOPF'S CLASSIFICATION THEOREM

Let M be a connected, orientable, closed n-manifold, and S^n the n-sphere ($n \geq 1$). Let $[M, S^n]$ denote the set of homotopy classes of continuous maps $f : M \to S^n$. As is well-known, the degree $\deg f$ of f induces the degree function $\deg : [M, S^n] \to \mathbb{Z}$, and H. Hopf [3] showed

Theorem 1.1. *The degree function* deg *is a bijection.*

There are many researches on the equivariant version of Hopf's classification theorem, i.e., the equivariant Hopf theorem (see [4, 2] etc). For example the following can be shown.

Theorem 1.2. *Suppose that a finite group G acts freely on S^n, $n \geq 1$.*

(1) *The degree function* $\deg : [S^n, S^n]_G \to \mathbb{Z}$ *is injective.*
(2) *The image of* deg *coincides with* $1 + |G|\mathbb{Z}$.

As a consequence, by setting $D([f]) = (\deg f - 1)/|G|$, we have the following equivariant Hopf type theorem.

Corollary 1.3. *The map* $D : [S^n, S^n]_G \to \mathbb{Z}$ *is a bijection.*

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2. Isovariant maps and isovariant homotopy classes

We consider an isovariant version of Hopf's classification theorem. Several results have been obtained in our previous works [8, 9]. In this article we present a generalization of them whose proof will be given in [10].

The notion of an isovariant map was introduced by Palais [11] in order to study a classification problem for orbit maps of G-spaces.

Definition. A (continuous) G-map $f : X \to Y$ between G-spaces is called G-*isovariant* if f preserves the isotropy subgroups, i.e., $G_{f(x)} = G_x$ for all $x \in X$. In other words, it is an equivariant map which is injective on each orbit of X. Similarly, if a G-homotopy $F : X \times I \to Y$ is G-isovariant, then it is called a G-isovariant homotopy.

Let $[X, Y]_G^{\mathrm{isov}}$ denote the G-isovariant homotopy set, i.e., the set of isovariant homotopy classes of G-isovariant maps from X to Y.

We investigate $[M, SW]_G^{\mathrm{isov}}$ for the following M and SW.

- M is a connected, *orientable*, closed *free* G-manifold (i.e., G acts freely on M).
- SW is a (unitary) representation sphere, i.e., the unit sphere of a *unitary* G-representation W. We assume that W is faithful (or equivalently G acts effectively on W).

We also assume the Borsuk-Ulam inequality:

$$\dim M + 1 \leq \dim SW - \dim SW^{>1}.$$

Here $SW^{>1}$ denotes the singular set of SW, i.e.,

$$SW^{>1} = \bigcup_{1 \neq H \leq G} SW^H.$$

As a convention, we set $\dim SW^{>1} = -1$ if $SW^{>1} = \emptyset$. The Borsuk-Ulam inequality is connected with a Borsuk-Ulam type theorem. Indeed, it appears in the following isovariant Borsuk-Ulam theorem.

Theorem 2.1. *Let M be a* $\mathrm{mod}\,|G|$ *homology sphere with free G-action ($G \neq 1$) and SW a representation sphere. If there is a G-isovariant map $f : M \to SW$, then*

$$\dim M + 1 \leq \dim SW - \dim SW^{>1}.$$

For other results on isovariant Borsuk-Ulam type theorems, see [12, 5, 6, 7]. Set

$$SW_{\mathrm{free}} = SW \setminus SW^{>1}.$$

Note that G acts freely on SW_{free}. Let $f : M \to SW$ be an isovariant map. By isovariance, it follows that $f(M) \subset SW_{\mathrm{free}}$. We may consider equivariant maps from

M to SW_{free}. In fact $[M, SW]_G^{\text{isov}}$ is identified with G-homotopy set $[M, SW_{\text{free}}]_G$:

$$[M, SW]_G^{\text{isov}} = [M, SW_{\text{free}}]_G.$$

In equivariant obstruction theory, the equivariant cohomology $\mathfrak{H}_G^*(M;\pi)$ plays an important role, where π is a $\mathbb{Z}G$-module. The equivariant cochain complex is defined by

$$C_G^*(M;\pi) := \mathrm{Hom}_{\mathbb{Z}G}(C_*(M);\pi),\ \delta := \mathrm{Hom}_{\mathbb{Z}G}(\partial).$$

Definition.

$$\mathfrak{H}_G^*(M;\pi) := H^*(C_G^*(M;\pi),\delta)$$

In our case π is taken as $\pi_q(SW_{\text{free}})$, and so we need to know the homotopy group of SW_{free}.

3. Topology of SW_{free}

The following proposition holds.

Proposition 3.1. *Let $d = \dim SW - \dim SW^{>1}$.*

(1) SW_{free} *is $(d-2)$-connected, i.e., $\pi_q(SW_{\text{free}}) = 0$ for $0 \le q \le d-2$.*

(2)

$$\begin{aligned}\pi_{d-1}(SW_{\text{free}}) &\cong H_{d-1}(SW_{\text{free}};\mathbb{Z})\\ &\cong \oplus_{H\in\mathcal{A}} H_{d-1}(S(W^H)^\perp;\mathbb{Z})\\ &\cong \oplus_{H\in\mathcal{A}}\mathbb{Z},\end{aligned}$$

where $\mathcal{A} = \{H \in \mathrm{Iso}\,(W) \mid \dim SW^H = \dim SW^{>1}\}$, and $(W^H)^\perp$ is the orthogonal complement of W^H in W.

Outline of Proof. Statement (1) follows from general position arguments.
(2): Note that $\dim S(W^H)^\perp = d-1$ for $H \in \mathcal{A}$. Using the Mayer-Vietoris exact sequence, one has

$$H_{d-1}(SW_{\text{free}};\mathbb{Z}) \cong \bigoplus_{H\in\mathcal{A}} H_{d-1}(S(W^H)^\perp;\mathbb{Z}) \cong \bigoplus_{H\in\mathcal{A}} \mathbb{Z}.$$

□

Remark. Note that $d \ge 2$ since W is unitary and faithful. When $d > 2$, the first isomorphism is obtained form the Hurewicz isomorphism. If $d = 2$, then $\dim M \le 1$ by the Borsuk-Ulam inequality, and so G must be cyclic. In this case, one also sees $\pi_1(SW_{\text{free}}) \cong \oplus_{(H)\in\mathcal{A}}\mathbb{Z}$.

Since G acts on SW_{free}, $\pi_{d-1}(SW_{\text{free}})$ and $H_{d-1}(SW_{\text{free}};\mathbb{Z})$ are regarded as $\mathbb{Z}G$-modules. For $H \in \mathcal{A}$, one can see that $gS(W^H)^\perp = S(W^{gHg^{-1}})^\perp$ for $g \in G$, and $gS(W^H)^\perp = S(W^H)^\perp$ iff $g \in NH$, the normalizer of H in G. Therefore we have

Lemma 3.2. *There are $\mathbb{Z}G$-isomorphisms*

$$\Psi : H_{d-1}(SW_{\mathrm{free}}; \mathbb{Z}) \to \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}[G/NH],$$

where $\mathcal{A}/G = \{(H) \mid H \in \mathcal{A}\}$, and

$$\Psi \circ h : \pi_{d-1}(SW_{\mathrm{free}}) \to \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}[G/NH],$$

where h is the Hurewicz homomorphism.

4. Equivariant obstruction theory

Set $\pi_{d-1} = \pi_{d-1}(SW_{\mathrm{free}})$ and $m = \dim M$. Let $f,\ g : M \to SW_{\mathrm{free}}$ be G-maps. Since SW_{free} is $(d-2)$-connected and $m \le d-1$, the equivariant obstruction class $\gamma_G(f,g)$ to the existence of a G-homotopy between f and g is defined in $\mathfrak{H}^m_G(M;\pi_m)$.

Remark. For $d = 2$, since π_1 is abelian, $\mathfrak{H}^*_G(M;\pi_1)$ is well-defined.

When $m \le d-1$, the equivariant obstruction class $\gamma_G(f,g)$ to the existence of a G-homotopy between f and g is defined in $\mathfrak{H}^m_G(M;\pi_m)$. Since $\mathfrak{H}^m_G(M;\pi_m) = 0$ when $m < d-1$, we have

Theorem 4.1. *If $m < d-1$, then $[M, SW]^{isov}_G = \{*\}$; namely, all isovariant maps from M to SW are isovariantly homotopic each other.*

Hereafter we assume that

$$m = d-1 \quad (m = \dim M,\ d = \dim SW - \dim SW^{>1}).$$

By equivariant obstruction theory, we have

Proposition 4.2. *The correspondence $[f] \mapsto \gamma_G(f_0, f)$ gives a bijection*

$$\gamma_{f_0} : [M, SW_{\mathrm{free}}]_G \to \mathfrak{H}^{d-1}_G(M;\pi_{d-1}),$$

where f_0 is a fixed isovariant map.

Remark. Since W is unitary and faithful, d is even and ≥ 2, and so $\dim M$ is odd.

Next we determine the equivariant cohomology group. Let $w : G \to \{\pm 1\}$ be the orientation homomorphism defined by setting, for $g \in G$,

$$w(g) = \begin{cases} +1 & \text{if } g \text{ acts orientation-preservingly on } M \\ -1 & \text{if } g \text{ acts orientation-reversingly on } M. \end{cases}$$

Let $\mathbb{Z}_w$ be the $\mathbb{Z}G$-module whose underlinying module is $\mathbb{Z}$ and the G-action is induced from the orientation homomorphism $w : G \to \{\pm 1\}$, i.e., $g \cdot k = w(g)k$. Let $K_w = \operatorname{Ker} w$ and

$$\mathcal{A}^+ = \{H \in \mathcal{A} \mid NH \le K_w\},$$

$$\mathcal{A}^- = \{H \in \mathcal{A} \mid NH \not\leq K_w\}.$$

We obtain the following result.

Theorem 4.3. *Under the assumption,*

$$\mathfrak{H}_G^{d-1}(M;\pi_{d-1}) \cong \bigoplus_{(H)\in\mathcal{A}^+/G} \mathbb{Z} \bigoplus \bigoplus_{(H)\in\mathcal{A}^-/G} \mathbb{Z}_2.$$

Consequently there is a one-to-one correspondence:

$$[M,SW]_G^{isov} \cong \bigoplus_{(H)\in\mathcal{A}^+/G} \mathbb{Z} \bigoplus \bigoplus_{(H)\in\mathcal{A}^-/G} \mathbb{Z}_2.$$

Proof. As seen before, $\pi_{d-1} \cong_G \bigoplus_{(H)\in\mathcal{A}/G}\mathbb{Z}[G/NH]$, and so

$$\mathfrak{H}_G^{d-1}(M;\pi_{d-1}) \cong \bigoplus_{(H)\in\mathcal{A}/G} \mathfrak{H}_G^{d-1}(M;\mathbb{Z}[G/NH]).$$

$\mathfrak{H}_G^{d-1}(M;\mathbb{Z}[G/NH]) \cong H^{d-1}(M/G;\{\mathbb{Z}[G/NH]\})$, where $\{\mathbb{Z}[G/NH]\}$ denotes the local coefficient system over M/G induced from the $\mathbb{Z}G$-module $\mathbb{Z}[G/NH]$. Using the Poincaré duality in local coefficients, we have

$$H^{d-1}(M/G;\{\mathbb{Z}[G/NH]\}) \cong H_0(M/G;\{\mathbb{Z}_w[G/NH]\}).$$

We then have $H_0(M/G;\{\mathbb{Z}_w[G/NH]\}) \cong$

$$\frac{\mathbb{Z}_w[G/NH]}{\langle a - w(g)a \mid a \in \mathbb{Z}[G/NH], g \in G\rangle} \cong \begin{cases} \mathbb{Z} & \text{if } NH \leq K_w \\ \mathbb{Z}_2 & \text{if } NH \not\leq K_w. \end{cases},$$

and so $\mathfrak{H}_G^{d-1}(M;\pi_{d-1}) \cong \bigoplus_{(H)\in\mathcal{A}^+/G}\mathbb{Z} \bigoplus \bigoplus_{(H)\in\mathcal{A}^-/G}\mathbb{Z}_2$. □

5. The multidegree and the isovariant Hopf theorem

We next introduce the multidegree of an isovariant map as a generalization of our previous definition. Set

$$SW_{\mathcal{A}^+\text{-free}} = SW \setminus \bigcup_{H\in\mathcal{A}^+} SW^H,$$

$$SW_{\mathcal{A}^-\text{-free}} = SW \setminus \bigcup_{H\in\mathcal{A}^-} SW^H.$$

Then

Lemma 5.1. *The inclusion*

$$i : SW_{\text{free}} \longrightarrow SW_{\mathcal{A}^+\text{-free}} \bigcap SW_{\mathcal{A}^-\text{-free}}$$

induces a $\mathbb{Z}G$-isomorphism

$$H_{d-1}(SW_{\text{free}}) \cong_G H_{d-1}(SW_{\mathcal{A}^+\text{-free}}) \oplus H_{d-1}(SW_{\mathcal{A}^-\text{-free}}).$$

Lemma 5.2. *Under identifying $H_{d-1}(SW_{\mathcal{A}^\pm\text{-free}};\mathbb{Z})$ with $\oplus_{(H)\in\mathcal{A}^\pm/G}\mathbb{Z}[G/NH]$,*

(1) *There exist integers* $d_H(f)$ *such that* $f_*^+([M]) = (d_H(f)\sigma_H)_{(H)\in\mathcal{A}^+/G}$, *where* $\sigma_H = \sum_{\bar{a}\in G/NH} w(a)\bar{a}$.
(2) $f_*^-([M]) = 0$.

This follows from the fact that, for any $g \in NH \setminus K_w$, g acts orientation-reversingly on M; on the other hand, $g \in G$ acts orientation-preserving on $S(W^H)^\perp$.

Remark. In $\mathbb{Z}_2$-coefficients, it also holds that $f_*^-([M]) = 0$.

Definition. The multidegree $\mathrm{mDeg}\, f$ of an isovariant map $f : M \to SW$ (or a G-map $f : M \to SW_{\text{free}}$) is defined by

$$\mathrm{mDeg}\, f = (d_H(f))_{(H)} \in \bigoplus_{(H)\in\mathcal{A}^+/G} \mathbb{Z}.$$

Clearly the multidegree is an isovariant homotopy invariant. The following is the main result.

Theorem 5.3. *Under the assumption,*
(1) *For any two G-isovariant maps* $f,\ g : M \to SW$,

$$\mathrm{mDeg}\, f - \mathrm{mDeg}\, g \in \bigoplus_{(H)\in\mathcal{A}^+/G} |NH|\mathbb{Z}.$$

(2) *Fix a G-isovariant map* $f_0 : M \to SW$. *For any* $\alpha \in \bigoplus_{(H)\in\mathcal{A}^+/G} |NH|\mathbb{Z}$, *there exists a G-isovariant map* $f : M \to SW$ *such that*

$$\mathrm{mDeg}\, f - \mathrm{mDeg}\, f_0 = \alpha.$$

(3) There are $2^{|\mathcal{A}^-/G|}$ G-isovariant homotopy classes with the same multidegree.
(4) In particular, if $\mathcal{A}^- = \emptyset$ (hence $\mathcal{A} = \mathcal{A}^+$), then

$$\mathrm{mDeg} : [M, SW]_G^{\mathrm{isov}} \to \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}$$

is injective.

By (1) of the above theorem, one can define $\mathrm{D}_{f_0}(f)$ by

$$\mathrm{D}_{f_0}(f) = \left(\frac{1}{|NH|}(d_H(f) - d_H(f_0))\right)_{(H)} \in \bigoplus_{(H)\in\mathcal{A}^+/G} \mathbb{Z},$$

where f_0 is a fixed isovariant map. Then we have the isovariant Hopf theorem.

Corollary 5.4. *If* $\mathcal{A}^- = \emptyset$, *then*

$$\mathrm{D}_{f_0} : [M, SW]_G^{\mathrm{isov}} \to \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}$$

is a bijection. In particular G acts orientation-preservingly on M, then D_{f_0} *is a bijection.*

Corollary 5.5. *Let M be a* mod $|G|$ *homology sphere with free G-action. Since we are assuming that* $\dim M = d-1$ *and W is unitary, it follows that* $\dim M$ *is odd. In this case, G acts orientation-preservingly on M, and hence we have*

$$D_{f_0} : [M, SW]_G^{isov} \cong \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}.$$

Remark. If $\dim M < d-1$, then $[M, SW]_G^{\text{isov}} = \{*\}$ as mentioned before. If $\dim M > d-1$, then $[M, SW]_G^{\text{isov}} = \emptyset$, since the isovariant Borsuk-Ulam theorem says that there is no isovariant map if $\dim M > d-1$.

Corollary 5.6. *Suppose that G is an abelian group. If the action on M is orientation-preserving, then $\mathcal{A} = \mathcal{A}^+$ and so*

$$[M, SW]_G^{isov} \cong \bigoplus_{H\in\mathcal{A}} \mathbb{Z}.$$

If the action on M is not orientation-preserving, then $\mathcal{A} = \mathcal{A}^-$ and so

$$[M, SW]_G^{isov} \cong \bigoplus_{H\in\mathcal{A}} \mathbb{Z}_2.$$

Remark. In the latter case, it follows that $\mathrm{mDeg}\, f = 0$ for any isovariant map.

6. Examples

Let $D_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{-1}\rangle$ be a dihedral group of order 2^n ($n \geq 3$). There are 3 conjugacy classes of subgroups of index 2: $D = \langle a^2, b\rangle \cong D_{2^{n-1}}$, $D' = \langle a^2, ab\rangle \cong D_{2^{n-1}}$, $C = \langle a^{2^{n-1}}\rangle \cong C_{2^{n-1}}$.

Let V_1 be a 2-dimensional irreducible representation of D_{2^n} such that C acts freely on SV_1. Set $W = sV_1$ for sufficient large s. Then one can see $\mathcal{A}/G = \{ ((b)), ((ab)) \}$ and $d = 2s$.

Let M_1, M_2, M_3 be $(2s-1)$-dimensional free D_{2^n}-manifolds whose K_w are D, D', C respectively. (Such D_{2^n}-manifolds exist for sufficiently large s.)

Example 6.1.

(1) $[M_1, SW]_{D_{2^n}}^{\text{isov}} \cong \mathbb{Z} \oplus \mathbb{Z}_2$.
(2) $[M_2, SW]_{D_{2^n}}^{\text{isov}} \cong \mathbb{Z} \oplus \mathbb{Z}_2$.
(3) $[M_3, SW]_{D_{2^n}}^{\text{isov}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Let V_2 be the 1-dimensional irreducible representation of D_{2^n} whose kernel is D. Next, we set $U = sV_1 \oplus V_2$. Then $\mathcal{A}/G = \{ ((b)) \}$ and $d = 2s$. Then we have

Example 6.2.

(1) $[M_1, SU]_{D_{2^n}}^{\text{isov}} \cong \mathbb{Z}$.
(2) $[M_2, SU]_{D_{2^n}}^{\text{isov}} \cong \mathbb{Z}_2$.
(3) $[M_3, SU]_{D_{2^n}}^{\text{isov}} \cong \mathbb{Z}_2$.

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